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Fundamental Torsional Frequency of a Class of Solid Wings

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This paper determines the fundamental torsional frequency of certain wings whose polar moment of inertia is proportional to the torsional rigidity at any section, as in a solid wing with a fixed t/c ratio. An inverse method of analysis is used in which the spanwise variation of rigidity and polar moment of inertia is related to an assumed spanwise variation of rotation in the fundamental torsional mode. The usefulness of this apparently hit-or-miss technique stems from the ease with which families of solutions may be generated. The results, which are augmented by a direct solution for wings with a constant taper, are expressed graphically in a form which relates the fundamental torsional frequency to the derived spanwise variation of the chord.

Nomenclature

c	= chord
C	= torsional rigidity of wing section, see Eq. (2)
f	= defined by Eq. (6)
F	= spanwise variation of chord, see Eqs. (7) and (12)
G	= shear modulus
J	= polar moment of inertia of wing per unit length
ℓ	= length of wing from supported end to tip
t	= wing thickness
T	= torque
x	= distance along wing measured from supported end
α,β	= parameters
θ	= rotation of wing section
ξ	$= x/\ell$
ω	= circular frequency
ω_0	= defined by Eq. (5)
Ω	$= \omega/\omega_0$

(See Fig. 1 for details.)

Introduction

THE torsional vibrations of a wing whose stiffness and inertia characteristics vary along the span seldom lend themselves to exact analysis and recourse must then be had to approximate methods. If a solid, unswept, and untapered wing is subjected to a constant torque T the twist $\mathrm{d}\theta/\mathrm{d}x$ is given by the equation

$$T = C\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{1}$$

where the torsional rigidity C is given by[1,2]

$$C = \frac{1}{3}G\int_{-\frac{1}{2}c}^{\frac{1}{2}c} t^3 \mathrm{d}y \tag{2}$$

where G is the shear modulus, c is the chord, and $t(y)$ is the thickness at a distance y from the midchord. There is also a warping of each cross section. When the torque varies along the span, as in a torsional vibration, the warping of cross sections also varies but the magnitude of the warping is less than that predicted by elementary theory because of the concomitant introduction of differential bending stresses in the wing. This feature results in an effective increase in the torsional rigidity as the wavelength of the vibration decreases. The concept of a torsional vibration, however, is a one-dimensional idealization in which it is tacitly assumed that wing sections rotate but otherwise maintain their shape. In reality, some transverse bending is introduced because of the distributed nature of the inertia loads, and this results in an effective decrease in the torsional rigidity as the wavelength of the vibration decreases. These two features thus tend to cancel each other and, indeed, for a wing of double-wedge (diamond) section they cancel each other exactly—in the sense that in any mode in which the deflections vary sinusoidally with x the true frequency, but not the precise mode shape, is the same as that predicted by elementary theory for all wavelengths.[3]

If the wing has a spanwise taper, whether constant or varying, the analysis for determining modes and frequencies is necessarily more complicated, and it is customary to adopt an elementary theory in which the wing can be treated in a one-dimensional manner, with Eqs. (1) and (2) maintaining their validity at each section. However, the resulting equations do not, in general, lend themselves to exact integration and recourse must then be had to approximate numerical techniques. The present paper obtains exact solutions by adopting an *inverse* approach in which the taper variation is determined in terms of a given spanwise variation of θ in the fundamental mode. This may seem to be very much a hit-or-miss approach but the technique is useful because of the ease with which solutions may be obtained. Indeed, it is shown that any given mode shape corresponds to a multiplicity of taper variations, each one associated with a given frequency.

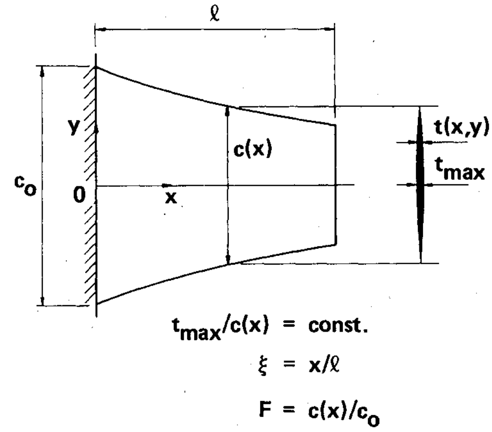

Fig. 1 Solid wing symmetrical about 0x axis.

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Analysis

Within the spirit of elementary theory, the rotation θ of a wing executing torsional vibrations with circular frequency ω satisfies the differential equation

$$\frac{d}{dx}\left(C\frac{d\theta}{dx}\right)+J\omega^2\theta=0 \tag{3}$$

where C is the torsional rigidity defined by Eq. (2) and J is the polar moment of inertia per unit length.

The boundary conditions for a wing supported at $x=0$ and free at $x=\ell$ are

$$[\theta]_{x=0}=0, \qquad \left[C\frac{d\theta}{dx}\right]_{x=\ell}=0 \tag{4}$$

and we note that if C and J do not vary with x the fundamental mode of vibration is given by

$$\theta \propto \sin\frac{\pi x}{2\ell}$$

with

$$\omega=\frac{\pi}{2\ell}\left(\frac{C_0}{J_0}\right)^{1/2}=\omega_0, \text{ say} \tag{5}$$

In what follows we consider solutions of Eq. (3) for wings whose torsional rigidity C and polar moment of inertia J vary with x in the same manner so that

$$C/C_0=J/J_0=f(\xi) \tag{6}$$

where $\xi=x/\ell$. This identity of variation occurs in a thin solid wing whose t/c ratio and section *shape* remains constant but whose chord, say, varies as

$$c(\xi)=c_0F(\xi) \tag{7}$$

so that

$$f(\xi)=\{F(\xi)\}^4 \tag{8}$$

Only the fundamental torsional mode is considered, and an inverse method of solution is adopted in which the variation of $f(\xi)$ is determined in terms of a given variation of $\theta(\xi)$. It is also convenient to introduce a nondimensional frequency term Ω defined by

$$\Omega=\omega/\omega_0 \tag{9}$$

so that Ω is unity for the untapered wing.

Equation (3) may now be written in the form

$$(f\theta')'+\tfrac{1}{4}\pi^2\Omega^2 f\theta=0 \tag{10}$$

where a prime denotes differentiation with respect to ξ.

Inverse Method of Solution

If $f\theta'$ is regarded as the dependent variable, Eq. (10) may readily be integrated to give

$$\ell n(f\theta')=-\tfrac{1}{4}\pi^2\Omega^2\int_0^\xi(\theta/\theta')\,d\xi+\text{const}$$

where the constant of integration must be chosen to ensure that $f=1$ at $\xi=0$, and hence

$$f=\left(\frac{[\theta']_{\xi=0}}{\theta'}\right)\exp\left\{-\tfrac{1}{4}\pi^2\Omega^2\int_0^\xi\left(\frac{\theta}{\theta'}\right)d\xi\right\} \tag{11}$$

Equation (11) determines the spanwise variation of torsional rigidity and polar moment of inertia corresponding to a given mode of vibration. The corresponding spanwise variation of the chord—and maximum wing thickness—is given by

$$F(\xi)=\{f(\xi)\}^{1/4} \tag{12}$$

Examples

Examples with Fixed Mode Shapes

Consider first the fundamental torsional mode of an untapered wing, Eq. (5). Substitution into Eq. (11) yields

$$f=(\cos\tfrac{1}{2}\pi\xi)^{\Omega^2-1} \tag{13}$$

and we thus derive a *family* of solutions of which the solution for the untapered wing, with $\Omega=1$, is a special case. If we introduce a parameter α for the index in Eq. (13), the solution can be written in a more traditional form, namely,

$$f=(\cos\tfrac{1}{2}\pi\xi)^{\alpha}, \qquad \Omega=(1+\alpha)^{1/2} \tag{14}$$

Other families of solutions may be obtained by choosing other forms for $\theta(\xi)$ which satisfy the boundary conditions of Eq. (4). For example, if $\theta\propto 2\xi-\xi^2$,

$$f=(1-\xi)^{(\frac{1}{4}\pi^2\Omega^2-1)}\exp\{\pi^2\Omega^2(2\xi-\xi^2)/16\} \tag{15}$$

if $\theta\propto 3\xi-\xi^3$,

$$f=(1-\xi^2)^{(\pi^2\Omega^2/12-1)}\exp(-\pi^2\Omega^2\xi^2/24) \tag{16}$$

if $\theta\propto\xi e^{-\xi}$,

$$f=(1-\xi)^{\pi^2\Omega^2/16-1}\exp\{\xi(1+\pi^2\Omega^2/16)\} \tag{17}$$

if $\theta\propto\xi e^{-\frac{1}{2}\xi^2}$,

$$f=(1-\xi^2)^{(\frac{1}{8}\pi^2\Omega^2-1)}\exp(\tfrac{1}{2}\xi^2) \tag{18}$$

and if $\theta\propto\xi e^{-\frac{1}{4}\xi^4}$,

$$f=\frac{e^{\frac{1}{4}\xi^4}}{(1-\xi^4)}\left(\frac{1-\xi^2}{1+\xi^2}\right)^{\pi^2\Omega^2/16-1} \tag{19}$$

All the above solutions are characterized by the fact that, except for a certain value of Ω, the function f tends to zero or infinity as $\xi\to 1$. Families of solutions with finite, nonzero tip values for f may, however, be readily achieved by the introduction of a parameter in the expression chosen for θ as shown below.

Examples with One-Parameter Mode Shapes

Consider first a typical example of how such mode shapes may be derived. A two-parameter expression for θ is first chosen, such as

$$\theta\propto\sin\alpha\xi e^{\beta\xi}$$

which satisfies the first boundary condition of Eq. (4). The second boundary condition is satisfied if

$$\beta=-\alpha\cot\alpha$$

so that a possible one-parameter mode shape that satisfies both boundary conditions is given by

$$\theta\propto\sin\alpha\xi\exp(-\alpha\xi\cot\alpha) \tag{20}$$

Substitution of Eq. (20) into Eq. (11) yields

$$f=\left(\frac{\sin\{\alpha(1-\xi)\}}{\sin\alpha}\right)^{k^2-1}\exp\{\alpha\xi(k^2+1)\cot\alpha\} \tag{21}$$

where $k=\pi\Omega\sin\alpha/2\alpha$.

The first term in the expression for f above is responsible for f tending to zero or infinity as $\xi\to 1$, but this may be avoided by taking $k=1$, i.e., if

$$\Omega=\frac{2\alpha}{\pi\sin\alpha}, \quad f=e^{2\alpha\xi\cot\alpha} \tag{22}$$

This particular solution is not novel; it was first obtained by Walker[4] using orthodox methods.

The following are further solutions with finite, nonzero tip values of f, and they are presented in the natural order f, Ω, θ rather than the derived inverse order:

$$f=(1+2\alpha\xi)^{(1-2\alpha)/2\alpha}\exp\{\xi(1-2\alpha)+\alpha\xi^2\}$$

$$\Omega=(2/\pi)(2\alpha+1)^{1/2} \tag{23}$$

$$\theta\propto\xi\exp\{\xi(2\alpha-1)-\alpha\xi^2\},$$

$$f=\left(1+\frac{\xi}{\alpha-1}\right)^{\alpha+1}\exp\left(\xi+\frac{\xi^2}{2\alpha}\right)$$

$$\Omega=\frac{2}{\pi}\left(\frac{\alpha-1}{\alpha}\right)^{1/2} \qquad \theta\propto\xi(\alpha-1+\xi)^{-\alpha} \tag{24}$$

$$f=\left(1+\frac{\xi^2}{2\alpha-1}\right)^{\alpha+1}\exp\frac{\xi^2}{2\alpha}$$

$$\Omega=\frac{2}{\pi}\left(\frac{2\alpha-1}{\alpha}\right)^{1/2} \qquad \theta\propto\xi(2\alpha-1+\xi^2)^{-\alpha} \tag{25}$$

$$f=\left(1+\frac{\alpha\xi(1+\alpha)}{2+\alpha}\right)^{4(1+\alpha)/\alpha^2}\exp-\left\{\frac{(2+\alpha)(1+\alpha^2)\xi}{\alpha}\right\}$$

$$\Omega=\frac{2}{\pi}(2+2\alpha+\alpha^2)^{1/2} \qquad \theta\propto\xi\left\{1-\xi\left(\frac{1+\alpha}{2+\alpha}\right)\right\}e^{\alpha\xi}, \tag{26}$$

$$f=\left(1+\frac{2\alpha\xi^2(1+2\alpha)}{3+2\alpha}\right)^{-(3+12\alpha+4\alpha^2)/4\alpha}e^{-\alpha\xi^2}$$

$$\Omega=\frac{2}{\pi}\{2+(1+2\alpha)^2\}^{1/2} \qquad \theta\propto\xi\left\{1-\xi^2\left(\frac{1+2\alpha}{3+2\alpha}\right)\right\}e^{\alpha\xi^2}, \tag{27}$$

$$f=(1+(1+3\alpha)\xi^2)^{-(4+12\alpha+3\alpha^2)/(2+\alpha)(1+3\alpha)}$$

$$\times\exp\left(-\frac{(2+3\alpha)\xi^2}{2(2+\alpha)}\right)$$

$$\Omega=\frac{2}{\pi}\left(\frac{10+15\alpha}{2+\alpha}\right)^{1/2} \qquad \theta\propto\xi\{1+\alpha\xi^2-(1+3\alpha)\xi^4/5\}, \tag{28}$$

$$f=(1+\alpha\sin^2\tfrac{1}{2}\pi\xi)^{-(4+\alpha)/(3+\alpha)}$$

$$\Omega=\left(\frac{1+\alpha}{1+\frac{1}{3}\alpha}\right)^{1/2} \qquad \theta\propto 3(4+\alpha)\sin\tfrac{1}{2}\pi\xi-\alpha\sin 3\pi\xi/2, \tag{29}$$

$$f=\left(\frac{1+2\cos\alpha\cos\alpha\xi}{1+2\cos\alpha}\right)^{3(\cot^2\alpha-1)/2} \qquad \Omega=\frac{2\alpha(2+\cos 2\alpha)^{1/2}}{\pi\sin\alpha}$$

$$\theta\propto 2\cos 2\alpha\sin\alpha\xi-\cos\alpha\sin 2\alpha\xi \tag{30}$$

$$f=(1+(1+2a)\xi)^{-2\alpha(3+6\alpha+2\alpha)^2/(2+\alpha)(1+2\alpha)^2}$$

$$\times\exp\left\{-\frac{(1+\alpha)\xi}{2+\alpha}\left(\xi-\frac{2\alpha}{1+2\alpha}\right)\right\}$$

$$\Omega=\frac{2}{\pi}\left(\frac{6(1+\alpha)}{2+\alpha}\right)^{1/2} \qquad \theta\propto\xi+\alpha\xi^2-\tfrac{1}{3}(1+2\alpha)\xi^3, \tag{31}$$

$$f=\left\{1+(1+\alpha)\xi+\alpha\xi^2\left(\frac{1+\alpha}{3+\alpha}\right)\right\}^{\tau}\left(\frac{1+k_1\xi}{1+k_2\xi}\right)^{\mu}\exp(\kappa\xi)$$

where

$$\tau=9(1+\alpha)/2\alpha^2, \quad \kappa=-(3+3\alpha+2\alpha^2)/\alpha$$

$$k_{1,2}=\frac{2\alpha(1+\alpha)}{(1+\alpha)(3+\alpha)\mp\lambda}, \quad \lambda=\{(1+\alpha)(3+\alpha)(3+\alpha^2)\}^{1/2} \tag{32}$$

$$\mu=-3(1+\alpha)(3+\alpha)^2/2\alpha^2\lambda$$

$$\Omega=\frac{2}{\pi}(3+3\alpha+\alpha^2)^{1/2}, \quad \theta\propto\xi\left\{1-\xi^2\left(\frac{1+\alpha}{3+\alpha}\right)\right\}e^{\alpha\xi}$$

Wing with Constant Taper

A direct solution for this case is now presented because of its practical importance. The ensuing analysis is simplified if x is now measured from the apex formed by extensions to the

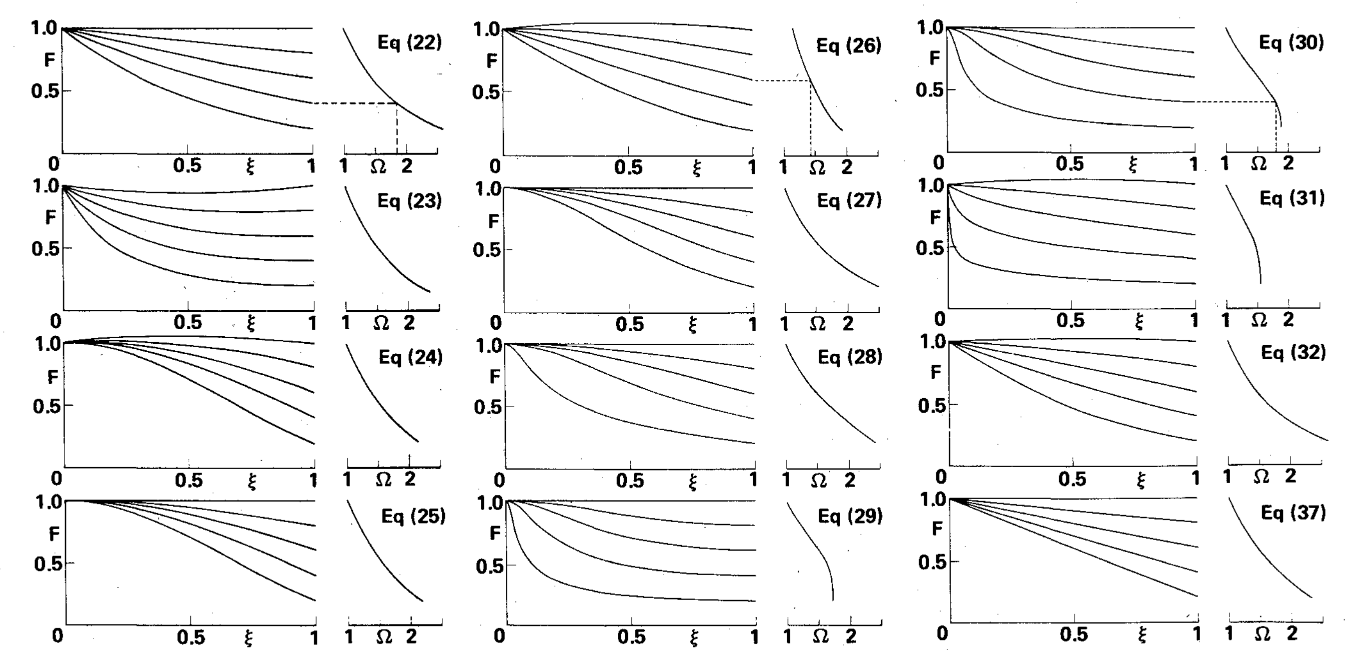

Fig. 2 Fundamental torsional frequency Ω for various solid wing planforms.

leading and trailing edges so that we can write

$$C=C_0(x/x_0)^4, \qquad J=J_0(x/x_0)^4 \tag{33}$$

where x_0, the distance from apex to wing root, equals $\ell/(1-\Delta)$, and Δ is the tip chord/root chord.

If now we introduce

$$z=\pi\Omega x/2\ell \tag{34}$$

it may be shown that Eq. (3) reduces to

$$\frac{d^2\theta}{dz^2}+\frac{4}{z}\frac{d\theta}{dz}+\theta=0 \tag{35}$$

Now Walker[4] has shown how this equation may be cast into the form of Bessel's equation by the introduction of $\mu=z^{3/2}\theta$. The resulting equation for μ has solutions that are Bessel's functions of order $\pm 3/2$, which can be expressed in terms of elementary functions. Thus it may be shown that the general solution of Eq. (35) may be expressed as

$$\theta=A(z^{-2}\cos z-z^{-3}\sin z)+B(z^{-2}\sin z+z^{-3}\cos z) \tag{36}$$

The boundary conditions of Eq. (4) now yield the following transcendental equation for the nondimensional frequencies Ω:

$$\tan\tfrac{1}{2}\pi\Omega=\frac{\tfrac{1}{2}\pi\Omega-\tfrac{1}{3}\Delta^2 z_0^3}{1+\Delta(1-\tfrac{1}{3}\Delta)z_0^2} \tag{37}$$

where $z_0=\pi\Omega/2(1-\Delta)$. Note that in contrast to the preceding inverse solutions, Eq. (37) yields the higher torsional frequencies in addition to the fundamental.

Presentation of Results

The results are applicable to unswept solid wings whose t/c ratio and section shape are fixed but whose chord, say, varies as

$$c(\xi)=c_0F(\xi) \tag{38}$$

where $F(\xi)=\{f(\xi)\}^{1/4}$.

The chordwise variations appropriate to Eqs. (22-32) and (37) are shown in Fig. 2. For each equation the five curves presented are those whose tip values are given by

$$F(1)=0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$$

These tip values also serve to identify the corresponding frequencies via the adjacent nomograph, as indicated by the broken lines on some of the figures.

Conclusions

The practical value of the inverse approach adopted in this paper is two-fold. First, it provides a vastly increased range of exact solutions, albeit confined to the fundamental torsional mode, against which the efficacy of approximate methods may be judged. Second, an inspection of Fig. 2 should enable the designer to estimate the fundamental torsional frequency of solid, unswept wings with any planform.

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